The Indispensabilist and the Autonomist

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§1: Terms

Let's start with some stipulation: Mathematical objects are abstract, having no spatial or temporal location and no causal powers. They exist necessarily and we know about them *a priori*, maybe, but anyway not through direct observation. Examples of mathematical objects include sets, numbers, spaces, knots, graphs, and structures.

Let's call mathematical platonism the claim that some such mathematical objects exist or that some mathematical claims (e.g. '$e^{\pi i} = -1$') are non-vacuously true. I won't argue here for mathematical platonism. Instead, I want to contrast the only two viable kinds of platonisms: indispensability platonism and autonomy platonism, argue that autonomy platonism is the better form of platonism, and say more about the *a priori* reasoning which could, if platonism were true, justify our mathematical beliefs.

§2: The Indispensability Argument

Indispensability platonism is platonism based on an indispensability argument. QI is my best estimation of what I take to be the strongest indispensability argument, Quine's.

QI
- QI1. We should believe only the theory which best accounts for our sense experience.
- QI2. If we believe a theory, we must believe in all of its ontological commitments.
- QI3. The ontological commitments of any theory are the objects over which that theory first-order quantifies.
- QI4. The theory which best accounts for our sense experience quantifies over mathematical objects.
- QIC. We should believe that mathematical objects exist.

According to the proponent of QI, statements like Coulomb's Law, CL, are parts of our best theory.

CL $\quad |\vec{F}| = k\,|q_1 q_2| / r^2$, where the electrostatic constant $k \approx 9 \times 10^9\ Nm^2/c^2$

When we examine carefully what CL says, regimenting it in our canonical language, we find that it comes out (at least at first approximation) something like CLR, asserting the existence of functions and real numbers, depending on the existence of physical objects, in this case charged particles.

CLR $\quad \forall x \forall y\{(Px\ \&\ Py) \rightarrow \exists f\,[f(q(x), q(y), d(x,y), k) = F]\}$
where $|F| = k\ |q(x)\ q(y)| / d(x,y)^2$

The argument QI is contentious in many ways. Some folks object to the naturalism at QI1. Some dispute the holism at QI2. Others deny Quine's criterion for ontological commitment at QI3. Perhaps most famously, many question the indispensability claim at QI4. Let's put all of those aside here, granting the argument QI, and let's focus on what I call the Unfortunate Characteristics of indispensability platonism. Most of these characteristics are well-known, so I will discuss them briefly, in order to contrast indispensability platonism with autonomy platonism.

First, since QI1 rules out alternative justifications for mathematical claims, the indispensabilist has no commitments to mathematical objects which are not required for our best empirical theories. Call this characteristic Restriction.

A second unfortunate characteristic, call it Ontic Blur, arises from the indispensabilist's inability to differentiate between abstract and concrete objects. Scientific theory does not support the distinction. The quantifier univocally imputes existence and the indispensabilist emphasizes continuity among the posits of a physical theory, from ordinary objects, through atoms, quarks, and maybe strings, through space-time points, to numbers and sets. There is no boundary between physical and mathematical objects as there is no boundary between real and merely instrumental posits.

As the abstract-concrete distinction erodes, so does the causal/non-causal distinction. Ordinary scientific theories do not include predicates for causal powers. So there is no reason to believe that the references of terms in the pure mathematical theorems included in an empirical theory are excluded from the causal realm.

The indispensabilist, further, has no grounds on which to say that mathematical objects exist necessarily. Mathematical objects are posited to account for our experience of a contingent world. In a different world, scientific theory would make different claims, and could require different objects. Suppose that electrical charge is a real property properly measured by real numbers. The indispensabilist thus alleges that the world contains continuous functions. Now suppose that in a different world, there are no continuous properties. In that world, says the indispensabilist, there are no continuous functions. Whether there are continuous functions in any particular world thus depends on contingent facts.

Similarly, the indispensabilist has no grounds to claim that mathematical objects are atemporal. If mathematical objects exist contingently, then there can be a time when they do not exist. If all continuous physical quantities were to be extinguished, the real numbers would likewise disappear.

Lastly, the indispensabilist's mathematical objects are, like concrete objects, known *a posteriori*. Indeed, many indispensabilists are motivated by a desire to avoid *a priori* epistemology.

The indispensabilist can deny these unfortunate characteristics, or introduce predicates into a canonical language to indicate the relevant classifications. But it is not clear how the indispensabilist may ground such assertions beyond mere fiat, undermining the basic tenet of the indispensabilist's argument that our scientific theories are designed to explain our sense experiences and not, as ends in themselves, mathematical phenomena.

The Unfortunate Characteristics

UC1 Restriction: The indispensabilist's mathematical commitments are only to those objects required by empirical science.
UC2 Ontic Blur: The indispensabilist can not support an abstract/concrete distinction.
UC3 Causality: The indispensabilist can not claim that mathematical objects are acausal.
UC4 Modal Uniformity: The indispensabilist's mathematical objects do not exist necessarily.
UC5 Temporality: The indispensabilist's mathematical objects exist in time.
UC6 Aposteriority: The indispensabilist's mathematical objects are known *a posteriori*.
UC7 Methodological Subservience: Any debate over the existence of a mathematical object will be resolved, for the indispensabilist, by the needs of empirical theory.

§3: Autonomy Platonism

The autonomy platonist claims that our beliefs in some mathematical claims are justified independently of the uses of mathematics in empirical scientific theories. She makes three claims in contrast to the indispensabilist.

Mathematical Evidence: Purely mathematical evidence, independent of sense experience, suffices to justify mathematical claims.

Theory Independence: Mathematical theories are independent of empirical theories, true or false regardless of the nature of the physical world. Mathematical theories are never refuted by empirical evidence.

Independence of Practice: Mathematical methods, including proof and intuition, are independent of empirical scientific methods.

Mathematical Evidence is a denial of the indispensabilist's claim that all experience is sense experience. Of course, there are sense experiences associated with any mathematical claim, like observations of symbols. But mathematical propositions are independent of any particular representation and thus of any particular sense experience. The evidence for the law of cosines holds even though there are no sensible triangles, and even though physical space is non-Euclidean. Its truth depends instead on its provability from axioms, perhaps, and on however we justify our beliefs in those axioms.

Theory Independence is the claim that mathematical theories are never refuted by empirical discoveries. Scientists use and discard equations and formulas. They refine their constants and improve their models. A scientist's abandonment of a mathematical tool is no evidence of any flaw, no sign of mathematical trouble.

It is difficult to delimit precisely empirical evidence from mathematical evidence, though particular cases can be easy to decide. Inductive evidence for a mathematical hypothesis, for example, is insufficient. Consider the strong inductive argument for Goldbach's conjecture that every even number greater than two can be expressed as the sum of two primes, called Goldbach pairs. By 2013, all even numbers through 4×10^{18} were verified to have Goldbach pairs, and the numbers of Goldbach pairs for each even number increases, though non-monotonically. Despite the capacity of this evidence to support our beliefs in Goldbach's conjecture, we do not consider the theorem proven.

Indispensabilists and autonomy platonists agree that mathematical proofs are sufficient as mathematical evidence. But indispensabilists believe that mathematical hypotheses need a further kind of evidence, one that comes from the application of a mathematical theory within science. Unless a mathematical theory is used in a scientific theory, indispensabilists look upon its proofs as recreational. The autonomy platonist denies that there is any evidence for a mathematical claim beyond its mathematical evidence. There is no external perspective on mathematics from which one can distinguish the true (i.e., applied) claims from the merely recreational ones.

§4: Intuition

Since autonomy platonism eschews appeals to sense experience in justifying mathematical beliefs, autonomy platonists must provide an alternative account. Bare appeals to the derivability of theorems from axioms are insufficient without an account of our knowledge of the axioms. Simple appeals to the immediacy and obviousness of the axioms are unsatisfying and misleading; some important axioms are neither immediate nor obvious. Our most secure mathematical beliefs may not be our best axioms. Moreover, mathematical theories are variously axiomatizable, with different axiomatizations having distinct virtues.

Still, the autonomy platonist holds that some mathematical propositions are to be taken as secure, if defeasibly so. Basic claims may be grasped intuitively. Such intuitions are experiences which can yield beliefs. These beliefs play an important, if not axiomatic, role in justifying our mathematical beliefs.

Kant invoked intuition in his account of mathematics, as did the intuitionists of the early twentieth century. By 'intuition', Kant means something particular to his epistemology: our cognitive faculty of receiving unconceptualized content. For Kant, we develop mathematics by reflecting on our pure forms of intuition, space and time. Such invocations of intuition are in the service of a conceptualist account of mathematics on which mathematical propositions are constructed in thought. For Kant and the intuitionists, mathematical objects are really mental objects. Such uses of intuition are not compatible with autonomy platonism. For the autonomy platonist, mathematical intuition is a capacity for acquiring, *a priori*, beliefs about abstract objects. We do not create the content of the belief about which we have an intuition; we grasp that content. I will not consider Kantian intuition further.

Gödel famously characterized a version of mathematical intuition which is consistent with autonomy platonism, taking it on analogy with perception: a non-inferential awareness, grasping, or understanding. In recent years, Gödel's view has been derided as incompatible with contemporary epistemology, untenable or desperate. The worries about access underlying these complaints are overstated; I will say more in §6

While there has been little recent work on the kind of mathematical intuition relevant to autonomy platonism, there is a useful variety of characterizations of philosophical intuition. George Bealer takes what he calls rational intuition to be a simple grasping, a conscious episode of seeming, and notes that intuition has a phenomenal character. As Bealer describes them, intuitions are not beliefs, commonsense opinions, judgments, the raising to consciousness of nonconscious background beliefs, guesses, hunches, or merely linguistic.

We can see the distinction between belief and intuition when we recognize that the set-theoretic comprehension axiom may seem true even though we have over-riding beliefs that show it to be false. The claims that intuitions are beliefs or commonsense opinions or judgments belies a category error. They may lead to beliefs or opinions, but they are not themselves beliefs.

Intuitions can not be identified with all of my nonconscious beliefs since I have many more non-conscious beliefs than I have even possible intuitions. And they can not be identified with raising nonconscious beliefs to consciousness because we often have intuitions which lead us to utterly new beliefs, as when I follow a proof of a new theorem.

Intuitions have different phenomenal character than guesses and hunches, which we cede when presented with contravening evidence. Intuitions are seemings, not guesses.

Rational and mathematical intuitions are closer to linguistic intuitions, as of the grammaticality of a sentence, though not all rational intuitions are linguistic. Linguistic intuitions regard words of a particular language; rational intuitions, as of 'if P then not-not-P', hold for any language.

These characterizations of intuition are adaptable to mathematical cases. On such views, intuitions are immediate inclinations to belief which take as their subjects concepts and objects, including modal properties, which are unavailable to sense experience.

Intuition mediates our recognition, through counterfactual reasoning, of the modal character of a mathematical proposition. When I develop a belief that an apple is red and ripe, I recognize that the apple could be green and may not be ripe. When I consider the sum of seven and five, I know that it must be twelve. This modal character of my mathematical beliefs arises from the nature of the objects of those beliefs, and we can see that mathematical intuition is *a priori* by considering the modal character of the beliefs we acquire intuitively.

That mathematical intuition is *a priori* should not be taken as entailing that the content of beliefs acquired by intuition are free from error. Mathematical claims ordinarily will be necessarily true, if true, and necessarily false, if false. But we are lamentably constructed so that we sometimes take false claims for true ones. The apriority of mathematical intuition is a recognition of the distinctness of the processes of acquisition and justification of mathematical beliefs from those for beliefs about concrete objects. It is not a guarantor of security for those beliefs. Intuition is not infallible. But when I have an intuition that p, I seek a theory which can accommodate p, or an explanation of why that intuition is compelling.

The role of intuition in forming mathematical theories is like the role of sense experience in forming empirical scientific theories: it provides particular data points which our best theories should accommodate maximally. In empirical cases, we have mechanisms to correct our false beliefs, including sense perception and considerations of theory construction. We might cede the belief that there is a pool of water in the distance either by approaching the mirage or by thinking more broadly about theories of light and reflection.

We have parallel mechanisms to correct false, but intuitive, beliefs. We cede mathematical beliefs in response to contrary intuitions or on the basis of broader theoretical considerations. In some simple forms, the axiom of choice is intuitively compelling. With the background of ZF, Choice is provably equivalent to the counter-intuitive well-ordering theorem. Moreover both Choice and its denial are consistent with the axioms of ZF. One or other intuition has to go; perhaps, even, the background ZF. We check our particular beliefs with further particular beliefs and systemic ones.

Intuition is often derided as a mysterious psychic ability. Michael Resnik complains that it is not "capable of scientific investigation or explanation" (Resnik 1997: 3-4.) Such a claim reveals a puzzling lack of ingenuity. People find some mathematical claims intuitive. We can ask about their intuitions, comparing reports and seeking theories to explain interpersonal consistencies and conflicts. We can use standard neuroscientific tools to see brain activity during an intuition.

Hilary Putnam insists that the autonomy platonist requires a dedicated brain structure for mathematical perception. Any account of mathematical reasoning must be consistent with neuroscience, but the connection may be more subtle than a dedicated region of the brain for mathematical perception. The claim that there are neural processes of mathematical perception would anyway be part of an empirical account of mathematics, not an apriorist one.

Such objections confuse mysterianism with philosophical puzzlement. Describing the nature of mathematical intuition is a philosophical and scientific challenge. For descriptions of intuition, we look toward neuroscience. For their justificatory status, we look toward mathematical epistemology.

The central concern for autonomy platonism about mathematical intuition is whether seemings are reliable to do proper justificatory work. We must explain the reliability of mathematical beliefs based on intuitions.

Part of the explanation of the general reliability of intuition, unsatisfying as it may be, is just that it is a brute, mundane fact about the abilities of mature reasoners that we reason well. Ordinary uses of intuition are unremarkably reliable largely because they often concern simple claims. We often just get *a priori* analysis of concepts right. We know that someone who is chasing is pursuing with the hope of catching, that a stroller is a walker, and that prime numbers other than two are odd. We are not born with utterly reliable mathematical intuitions. Our abilities require training which hones our skills. Our early perceptual and small motor skills are lousy at first too; we mature. Philosophers often debate at the boundaries of our knowledge of concepts, becoming distracted by errors at the fringes of our abilities. But ordinary mathematical reasoning is mundanely successful.

While intuition is sometimes portrayed as a mystical or non-natural ability, it is an ordinary aspect of our reasoning which must be included in any plausible mathematical epistemology. Our mathematical intuitions are intricately interwoven with our beliefs about mathematical theories. A broader account of the reliability of intuition must thus take all mathematical reasoning into account.

§5: From Mathematical Intuition to Mathematical Theory

A fallible mathematical intuition yields some mathematical beliefs. But intuitiveness does not account for the breadth and security of our mathematical theories. We check, systematize, and extend our beliefs. We compare different systematizations for their mathematical virtues and their more-general theoretical characteristics.

These processes of refining and improving mathematical beliefs are just the natural and well-refined methods of mathematics. Striving for balance between intuition and other mathematical methods is elsewhere known as seeking reflective equilibrium. We weigh our intuitive apprehensions of basic mathematical facts against the systematizations of our mathematical knowledge. Intuitions are constraints on the system-building and the systems are constraints on our estimations of our intuitions. Bertrand Russell describes the process neatly.

> When pure mathematics is organized as a deductive system - i.e. as the set of all those propositions that can be deduced from an assigned set of premises - it becomes obvious that, if we are to believe in the truth of pure mathematics, it cannot be solely because we believe in the truth of the set of premises. Some of the premises are much less obvious than some of their consequences and are believed chiefly because of their consequences (Russell 1924: 325).

We hold some intuitive mathematical claims: simple arithmetic facts, some rudimentary set-theoretic statements, maybe Hume's Principle. We organize and regiment those claims, both to see if they are consistent and to make connections with other mathematical theories. We find unifying claims among various theories and refine our axiomatizations. We balance our formal theories with our particular beliefs, adjusting the axioms as they fit the theorems, giving up some intuitive principles in order to achieve elegant systematizations.

Bishop Berkeley demurs, presenting an early version what one might call the circularity objection.

> I say that in every other science men prove their conclusions by their principles, and not their principles by the conclusions. But if in yours you should allow your selves this unnatural way of proceeding, the consequence would be that you must take up with induction, and bid adieu to demonstration. And if you submit to this, your authority will no longer lead the way in points of reason and science (Berkeley, *Analyst*, §19).

For similar reasons, Hartry Field calls indispensability platonism the only non-question-begging version of platonism (Field 1980: 4). The autonomy platonist argues that we should believe that some mathematical theories are true on pure mathematical evidence. But we should believe that mathematical evidence is reliable because it is consistent with our mathematical theories. We believe that '7+5=12' is true in part because we believe that it follows from the Dedekind/Peano axioms of arithmetic. But we believe those axioms in part because they yield true propositions of arithmetic.

This Circularity Objection to autonomy platonism is related, at least formally, to an accusation of mysterianism or theology. The psychic tells you to believe the crystal ball; the theist tells you to believe in scripture, and the autonomy platonist tells you to believe in mathematical theories. For the indispensabilist, evidence for mathematical claims is empirical. For the autonomy platonist, evidence is dangerously ungrounded.

A parallel accusation of circularity can be levied against the empiricist: we believe that there are trees in part because our senses tell us that there are trees. We believe that sense experience is accurate in part because it tells us that there are things like trees, which we take as manifest. The accusation of circularity, as we are considering it, is supposed to hold against the autonomy platonist and not the indispensability platonist. Such an accusation can hold only if we have independent justification for our beliefs in our empirical theories.

One obvious response to the charge of circularity for empirical theories is to claim that the evidence of sense experience is somehow basic or secure from error. The logical empiricists, for example, tried to ground scientific theories in sense data, which they took to have a special status secure from error.

One problem with logical empiricism is that it takes observations as evidence when they are all laden with theory. As Quine saw, any particular claim can be held true as long as we make appropriate adjustments to our background theory. Quine showed that confirmation holism better depicts the relations among theories and observations than the logical empiricist's dogma of reductionism. We start our theorizing with the tentative evidence of sense experience. As we add experiences, we construct increasingly plausible theories to account for them. We balance our tentative observations with our tentative theories, working in both directions to formulate the most attractive, comprehensive, conservative, and powerful system of beliefs that we can. We use our evidence to support our theories and we use our theories to judge and predict our evidence. To deny the legitimacy of sense experience because of the circularity of such reasoning is self-defeating skepticism.

What separates the autonomy platonist from the indispensability platonism is the not form of argument, then, but the nature of the evidence. For the indispensabilist, all evidence is sense evidence. For the autonomy platonist, purely mathematical evidence is no less secure, no less reliable, and no less respectable.

As we move from rough intuitions to precise mathematical theories, we are guided by three cognitive tools: our ability to recognize consistency, our inferential powers, and our mathematical intuition. The former guides the mathematician categorically, especially where she invokes rigorous formal systems. Our inferential tools are both formal and intuitive. The latter guides the mathematician where the former two fail.

In assessing the balance between axiomatizations and particular mathematical claims, intuition again guides us fallibly. We give up some basic claims which appear intuitive, like Euclid's parallel postulate. We find that certain systematizations better organize mathematical phenomena than others. We balance strength and elegance. We seek reflective equilibrium between axioms and particular theorems, guided by consistency and our estimations of the theoretical virtues.

The process of seeking reflective equilibrium in mathematics embraces ordinary, natural descriptions of mathematical methods. There are deep and interesting questions about those methods, but the procedure I have described briefly is general enough to be adapted naturally to whatever gets properly called mathematics. That's autonomy, and its role in the indispensabilist's view is sadly anemic.

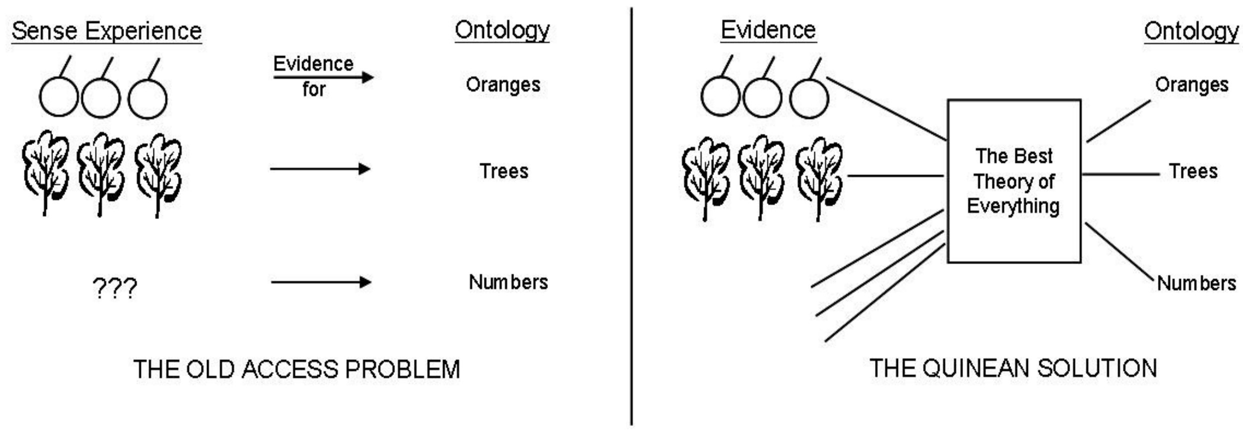
§6: On What Mathematical Objects There Are

Despite its appeals to mathematical intuitions, autonomy platonism is relatively mundane, adopting ordinary mathematical methods. While philosophers sometimes bristle at invocations of intuition, mathematicians are generally familiar and comfortable with the phenomenon. But platonism is a contentious view largely because of its acceptance of abstract objects, inaccessible to the sense receptors of concrete things. The autonomy platonist must provide a satisfying account of our knowledge of mathematical objects.

This central challenge for platonism is known as the access problem. To demand access is to demand that we correlate objects we believe to exist with particular perceptions. The traditional empiricist requires lines of access from, say, a tree to my eyes, to my brain, to my beliefs about the tree. She seeks roughly parallel lines to account for beliefs about all objects.

One approach to the access problem for mathematics involves positing an ability to perceive abstract objects, as Plato, Descartes, and Gödel (perhaps) did. But we do not perceive abstract objects. Any implication that we do must be strictly metaphorical. Any such metaphor must be explained in literal terms.

A better solution to the access problem comes from Quine's work, independent of the indispensability argument, in his view of ontology as a system of posits. In lieu of direct accounts of access to objects, Quine isolates evidence, on one side of a theory, from ontology, on the other. Between them stands a theory which must be consistent with the evidence and attractive and parsimonious. For



Quine, the access problem is a non-starter since *no* ontological commitments are mere consequences of perception. Our perceptions are constraints on the construction of our theories. Our ontology is the result of interpreting and understanding those theories.

Quine's approach, consistent with autonomy platonism, eliminates questions of access. What there is, is what our best theories say there is. The difference between the autonomy platonist and the indispensabilist is on the question of which theories we should believe. The indispensabilist says that mathematical theories, considered independently of empirical scientific theories, fail to compel our belief. Only when they are invoked to account for our sense experiences do they cease being merely recreational. The autonomy platonist argues that mathematical theories themselves are proper objects of belief whether or not they are used in physical science.

So we should believe our mathematical theories because they are the reliable results of systematizing our mathematical intuitions. Beliefs in mathematical objects follow. We use intuition to justify our mathematical beliefs, but not as a special faculty of perception of mathematical objects. Our knowledge of mathematical objects is just a result of interpreting our best mathematical theories, taking mathematical objects to be posits of mathematical theories in the same way that electrons and cats are posits of scientific theories.

§7: Conclusion

When we reflect on our methods in philosophy and in science, we find that they are generally attempts to align particular beliefs with general claims. We abandon some particular claims for the benefits of elegant and broad systematizations. We extend our systematizations to comprehend further particular claims. There is nothing mystical or magical about this process.

That we hold our claims fallibly does not entail that we are debarred from making claims about the necessity of mathematical truths, when true, or that our reasoning in proofs and in the understanding of particular claims, is *a priori*. We can thus defend mathematical platonism, in close to a traditional forms, without abandoning our ordinary conceptions of ourselves.

Still, traditional platonism has been attractive to many folks for its securing of our mathematical beliefs. If our intuitions are fallible, our theories are guided by no more secure method, and our ontology arises from interpreting our theories, it seems that the traditional security of mathematical reasoning to abstract objects is lost. So be it. The Humean condition is the human condition, even in mathematics. Claims of traditional platonists to foundational security were unfounded.

Still, a glib dismissal of old-fashioned foundationalism goes no distance to answering the worry about whether autonomy platonists can justify their mathematical beliefs. The method of reflective equilibrium on which autonomy platonism is founded seems, like coherentist epistemologies generally, to be liable to be untethered to mathematical truth. The method threatens to undermine the justificatory role that an epistemology for mathematics is supposed to provide. This challenge for the autonomy platonist can be met, but not here.